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RISK ASSESSMENT OF MANY-COMPONENT SYSTEMS WITH SPECIFIED SPARE PROVISIONING POLICIES

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ABSTRACT

Markov models are proposed for risk assessment of many-component systems with specified spare provisioning policies. Unlike in existing models, each component is either repaired or discarded on failure. The probability of component shortage (risk probability) is calculated. The results will be useful for application in systems with groups of identical instruments such as transformers of the electric power system, valves and small pumps in piping systems of generating stations.

1. INTRODUCTION

Spare provisioning models and their risk assessments have been widely reported in the literature/1-6/. In these models, the many-component systems under consideration are assumed to exhibit only one of the following two characteristics:

(i) Any failed component will be repaired.

(ii) Any failed component will be discarded.

In practice, however, there are repairable systems in which components are also susceptible to discarding on failure. The decision to discard is based on the level of damage to the components, repair time and cost considerations. Examples of such components are the potential transformers and the current transformers of the electric power system, and the valves and small pumps in piping systems of the generating stations.

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In this report, risk assessment models are developed for systems with components normally repaired, but also susceptible to discarding on failure. The system under consideration consists of a finite (but large) number of identical components in service. When one of these components fails, it is immediately replaced by a spare component (if one is available). The failed component may be either discarded or repaired. On completion of repair, it will be kept as a spare component. A finite number of repair teams is available.

If the component level (that is, the total number of components in the system excluding discarded components) falls below a specified value, an order for the purchase of a fixed batch of spare components will be made. This order will arrive after some random epoch of time. Such replenishment policy is known as the (s,q) policy in inventory theory /3,5,6/. Figure 1 shows the variation of component level as a function of time.

2. NOMENCLATURE

Component level:	total number of components in the system excluding discarded components. Therefore, it is the total number of operating components, spare components and failed components (which are not discarded) in the system.
Order level:	value at which the component level has fallen, resulting in a purchase order for a batch of spare components.
Order quantity:	the quantity of spare components to be purchased when the order point is reached.
Lead time:	time interval between placement of purchase order and receipt of spare components.
Discarding probability:	probability that a failed component will be discarded.
Risk probability:	probability of shortage of one or more components
Traffic intensity:	ratio of the rate of component failures (repairable) to the repair rate.
Discarding intensity:	ratio of the average number of failed components discarded during the lead time to the order quantity.

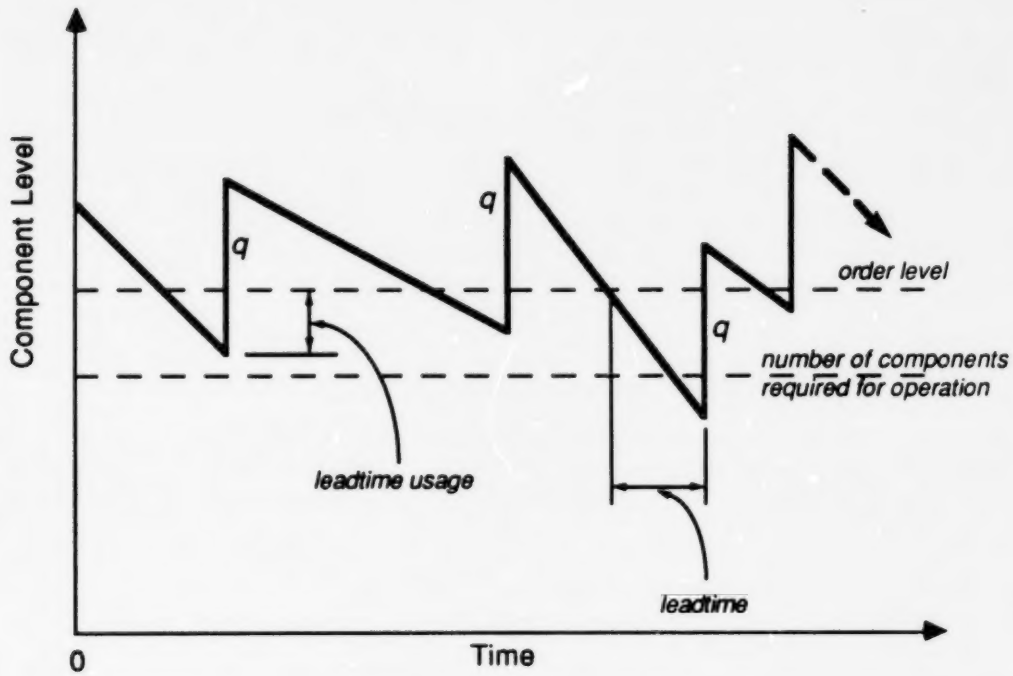

FIGURE 1

COMPONENT LEVEL AS A FUNCTION OF TIME
(q = order quantity)

3. ASSUMPTIONS

1) The system is in the steady state.

2) The times to failure for each component in operation are exponentially distributed.

3) Components will not fail in reserve state.

4) Failed components will be replaced with spare components (if any). The replacement times are assumed to be negligible.

5) Each failed component has an identical constant probability of being discarded. If it is not discarded, it will undergo repair. On completion of repair, it becomes a spare component.

6) A finite number of repair teams is available. Repair is on a "first-come, first-served" basis.

7) Repair times are exponentially distributed.

8) Whenever the component level falls to the order level, an order for a batch of spare components will be triggered. The lead times are also exponentially distributed.

4. NOTATIONS

N	-	total number of components required for operation
c	-	total number of repair teams
s	-	order level
q	-	order quantity
k	-	truncation parameter in the model
α	-	probability that a failed component will be discarded
λ^{-1}	-	mean time to failure for each component in operation
μ^{-1}	-	mean time to repair a failed component
β^{-1}	-	mean lead time
ρ	-	traffic intensity

ψ	-	discarding intensity
m	-	number of operating components and spare components in the system, $N - k \le m \le N + s + q$
j	-	number of components awaiting or under repair, $j \le N + s + q - m$
l	-	component level, $l = m + j$
P(m,j)	-	steady state probability that the system is in state (m,j)
P(m)	-	steady state probability that the system is in state m
U	-	risk probability

5. MODEL 1

In this section, a model is developed to predict the state probabilities of a many-component system with a supply of spare components. It is assumed that the component level never falls below the order quantity.

The full system of simultaneous equations describing the state probabilities can be easily derived. However, the computation effort needed for solving a system of simultaneous equations with a large number of unknowns can be tremendous. Hence, the system of equations is truncated by setting the state probabilities to zero whenever the number of operating components falls below N-k.

5.1. Model Formulation

Define:

$$\lambda_m = m\lambda \quad N-k \le m < N$$
$$= N\lambda \quad m \ge N$$
$$= 0 \quad \text{otherwise}$$

$$\mu_j = j\mu \quad 1 \le j \le c$$
$$= c\mu \quad j \ge c$$
$$= 0 \quad \text{otherwise}$$

$$\gamma_{m,j} = \lambda_m + \beta + \mu_j \quad m + j \le N + s,\ m \ge N-k,\ j > 0$$

$$= \lambda_m + \mu_j \qquad m + j > N + s,\ m \geq N-k,\ j > 0$$
$$= \lambda_m + \beta \qquad m \leq N + s,\ j = 0$$
$$= \lambda_m \qquad m > N + s,\ j = 0$$

The difference equations for the steady state probabilities P(m,j) are:

$$P(m,j) = 0 \quad (m < N\text{-}k,\ j \geq 0) \ \text{ or } (m + j > N + s + q) \tag{1}$$

$$\gamma_{m,0}P(m,0) = \lambda_{m+1}\alpha P(m+1,0) + \mu_1 P(m-1,\ 1) + \beta P(m-q,0) \qquad N-k \leq m < N+s+q \tag{2}$$

$$\gamma_{N+s+q,0}P(N+s+q,0) = \mu_1 P(N+s+q-1,1) + \beta P(N+s,0) \tag{3}$$

$$\gamma_{m,j}P(m,j) = \lambda_{m+1}\alpha P(m+1,j) + \lambda_{m+1}(1-\alpha)P(m+1,j-1) + \mu_{j+1}P(m-1,j+1) + \beta P(m-q,j) \tag{4}$$
$$N-k \leq m < N+s+q-1,$$
$$1 \leq j < N+s+q-m.$$

$$\gamma_{m,N+s+q-m}P(m,N+s+q-m) = \lambda_{m+1}(1-\alpha)P(m+1,N+s+q-m-1) + \mu_{N+s+q-m+1}P(m-1,N+s+q-m+1) + \beta P(m-q,N+s+q-m) \tag{5}$$
$$N-k \leq m < N+s+q$$

$$\sum_{m=0}^{N+s+q} \sum_{j=0}^{N+s+q-m} P(m,j) = 1 \tag{6}$$

The conditions for the system to reach and remain in steady state are:

(1) The rate of component failures (repairable) is less than the repair rate. Mathematically, this is written as

$$\rho = \frac{N\lambda\,(1-r)}{c\mu} < 1\,.$$

(2) The average number of discarded components during the lead time is less than the order quantity. That is,

$$\psi = \frac{N\lambda\,\alpha}{q\beta} < 1 .$$

It is obvious that a very reliable system is associated with low traffic intensity (e.g. $\rho < 0.1$). Further improvement in reliability can be achieved by lowering the discarding intensity ψ.

As the discarding probability α increases, the traffic intensity descreases while the discarding intensity increases given that the other parameters remain constant.

As the mean replenishment lead time approaches zero, the discarding intensity also approaches zero. However, the traffic intensity will remain unaffected.

A special case of the above model is the machine interference model with N repairable components, s spares and c repair teams /4/. This is obtained by setting $\alpha = 0$, $\beta = 0$ and $q = 0$ in equations (1 to 6).

5.2. Risk Probability

This is the steady state probability that the system does not meet its full requirement, that is, there is a shortage of at least one component. Mathematically, this is given by

$$U = 1 - \sum_{m=N}^{N+s+q} \sum_{j=0}^{N+s+q-m} P(m,j)$$

where the state probabilities P(m,j) are obtained by solving the system of equations (1-6) using a numerical approach such as Gauss-Seidel iteration.

6. MODEL 2

In this model, the repair time of each failed component (which is not discarded) is assumed to be negligible in comparison with operating time. The justification for this model will be discussed in Section 7.

6.1. Formulation

The steady state probability equations describing the model are:

$$\beta P(0) = \lambda_1 \alpha P(1) \qquad (7)$$

$$(\lambda_m\alpha+\beta)P(m)=\lambda_{m+1}\alpha P(m+1) \qquad 1\le m\le q-1 \tag{8}$$

$$(\lambda_m\alpha+\beta)P(m)=\lambda_{m+1}\alpha P(m+1)+\beta P(m-q) \qquad q\le m\le N+s \tag{9}$$

$$\lambda_m\alpha P(m)=\lambda_{m+1}\alpha P(m+1)+\beta P(m-q) \qquad N+s+1\le m\le N+s+q-1 \tag{10}$$

$$\lambda_{N+s+q}\alpha P(N+s+q)=\beta P(N+s) \tag{11}$$

$$\sum_{m=0}^{N+s+q} P(m)=1 \tag{12}$$

For the system to reach and remain in steady state, the following condition must be satisfied:

$$\psi=\frac{N\lambda\alpha}{q\beta}<1.$$

Note that a spare provisioning model with non-repairable components is obtained by setting $\alpha = 1$ in the above system of equations. This is actually the (s,q) inventory model /5/.

6.2. Risk Probability

The risk probability is given by

$$U=1-\sum_{m=N}^{N+s+q} P(m)$$

where the state probabilities P(m) are obtained by solving the system of equations (7-12). The solution approaches are given in Section 6.3.

6.3. Methods of Solution

In this section, solution approaches are proposed. Equations (7 to 12) can be rewritten to give the following:

$$P(m) = \frac{\omega}{\lambda_1} P(0) \qquad m = 1$$

$$= \frac{\omega}{\lambda_1} \prod_{i=2}^{m} \left[\frac{\lambda_{i-1}}{\lambda_i} + \frac{\omega}{\lambda_i} \right] \qquad 2 \le m \le q$$

$$= \left[\frac{\lambda_{m-1}}{\lambda_m} + \frac{\omega}{\lambda_m} \right] P(m-1) - \frac{\omega}{\lambda_m} P(m-q-1) \qquad q < m \le N+s+1$$

$$= \frac{\lambda_{m-1}}{\lambda_m} P(m-1) - \frac{\omega}{\lambda_m} P(m-q-1) \qquad N+s+2 \le m \le N+s+q$$

where $\omega = \beta/\alpha$.

6.3.1. Exact Recursive Solution

Let $A(m) = P(m)/P(0)$, $m = 0, 1, 2, ..., N + s + q$. The following recursive scheme is proposed to evaluate P(m).

Step 1: Observing that, by definition, $A(0) = 1$, compute A(m) for $m = 1, 2, ..., N + s + q$.

Step 2: Compute P(0) using the normalizing equation:

$$P(0) = \left[1 + \sum_{m=1}^{N+s+q} A(m) \right]^{-1}.$$

Step 3: Compute P(m) using the equation:

$$P(m) = A(m)P(0), \quad m = 1, 2,...., N + s + q.$$

6.3.2. Approximate Recursive Solution

Since the system is highly reliable, it is reasonable to truncate the system of equations so that $P(m) = 0$ for $0 \le m < N-k$.

Let $B(m) = P(m)/P(N-k)$ for $m = 0, 1, 2, \ldots, N + s + q$. The following recursive scheme is proposed to evaluate P(m).

Step 1: Observing that by definition, $\{B(m)=0,\ 0 \le m < N-k\}$ and $B(N-k) = 1$, compute B(m) for $m = N-k+1, N-k+2, \ldots, N + s + q$.

Step 2: Compute P(N-k) using the normalizing equation:

$$P(N-k) = \left[1 + \sum_{m=N-k+1}^{N+s+q} B(m) \right]^{-1}$$

Step 3: Compute P(m) using the equation:

$$P(m) = B(m)P(N-k), \quad N-k+1 \le m \le N+s+q$$

7. RESULTS

It is of interest to find out how the risk probability depends on parameters such as the order level, order quantity, component's mean time to failure, component's mean repair time, discarding probability and mean lead time.

In the case of model 1, the time required for computing the risk probability is also dependent on the chosen value of the truncation parameter k. In general, the accuracy of the computed risk probability increases as k increases. For more reliable systems, it is reasonable to assume a smaller value for k. Figure 2 shows the effect of the truncation parameter on the accuracy of the computed risk probability. For the particular example in Figure 2, it is good enough to set $k = 30$.

The risk probability decreases as the order quantity increases. This is shown in Figure 3 for values of order level varying from 5 to 9. However, for systems with negligible repair times of failed components, it is not necessary to know the individual values of discarding probability α and mean lead time $(1/\beta)$. Instead, it is sufficient to know the value of the ratio β/α. The results are plotted in Figure 4.

As expected, the risk probability increases as the mean repair time increases. This is shown in Figure 5 for values of component's mean time to failure varying from 1200 days to 1800 days. For this example, the risk probability remains almost constant if the mean repair time is less than 5 days. This is particularly true for mean time to failure exceeding 1500 days. Hence, it is reasonable to use the simple model in Section 6 if the mean repair time is very small relative to mean time to failure.

The discarding probability has a significant effect on the risk probability as discussed in Section 5.1. Figure 6 shows the risk probability as a function of discarding probability for total number of repair teams c = 1 and c = 2. For the particular example on c = 1, the risk probability decreases as the discarding probability increases to 0.5. Thereafter, the risk probability increases. However, for c = 2, the risk probability behaves differently. It increases with respect to any increase in discarding probability.

As expected, the risk probability decreases as the mean lead time decreases. The results are shown in Figure 7. Note that the risk probability may not approach zero as the mean lead time becomes zero. This is due to the effect of traffic intensity as discussed in Section 5.1.

8. CONCLUSIONS

(1) Two Markov models have been developed for risk assessment of many-component systems with specified spare provisioning policies. Unlike existing models, each component is either repaired or discarded on failure. The models generalize the traditional machine interference models /4/ and the classical (s,q) inventory models /3,5,6/.

(2) Model 1 is a fairly general model. However, it involves the solution of a system of simultaneous linear equations with a large number of unknowns. This can be computationally time-consuming.

(3) Model 2 can be easily derived from model 1 by setting the mean repair time to be zero. Its solution is very simple and fast. Despite its assumption on zero repair times, it has been found to be accurate in predicting the system's risk probability as long as the component's mean repair time is very small compared with its mean time to failure.

(4) The risk probability is found to be dependent on system parameters such as the order level, order quantity, component's mean time to failure, component's mean repair time, discarding probability and mean lead time. Hence, a specified risk probability can be obtained from different combinations of these parameters. This will be useful in finding the lowest operating cost for the system subject to a specified risk criterion. Such risk-cost analysis should be considered in future studies.

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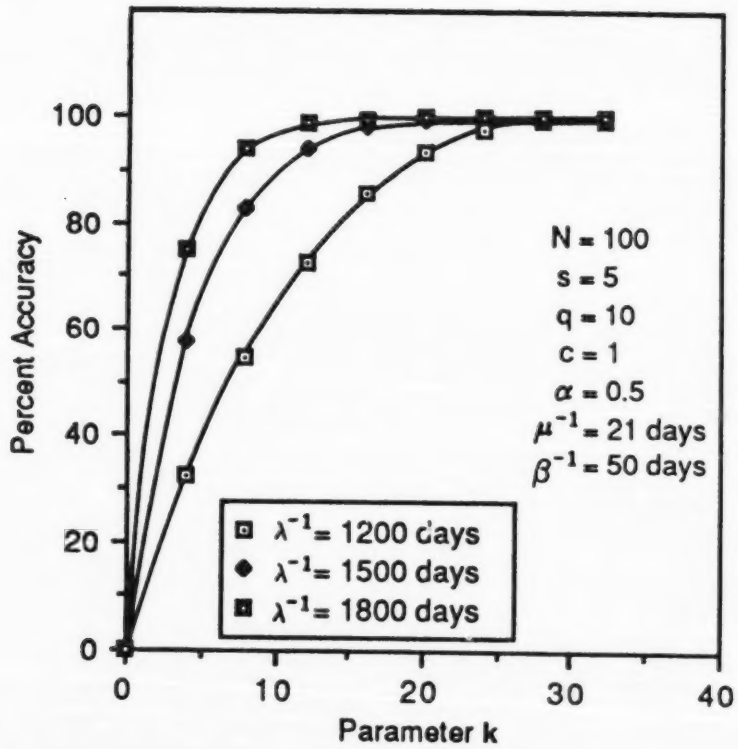

FIGURE 2

ACCURACY OF COMPUTED RISK PROBABILITY AS A FUNCTION OF MODEL TRUNCATION PARAMETER k

CAZ DISK
SIM-FIG-2

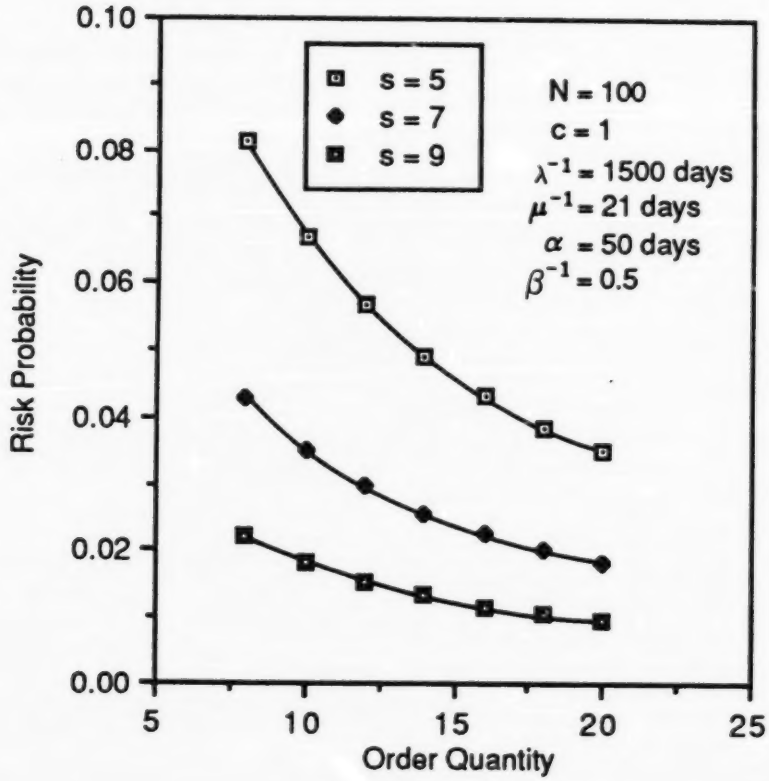

FIGURE 3

RISK PROBABILITY AS A FUNCTION OF ORDER QUANTITY
(indicating the effect of order levels)

CAZ DISK
SIM-FIG-3

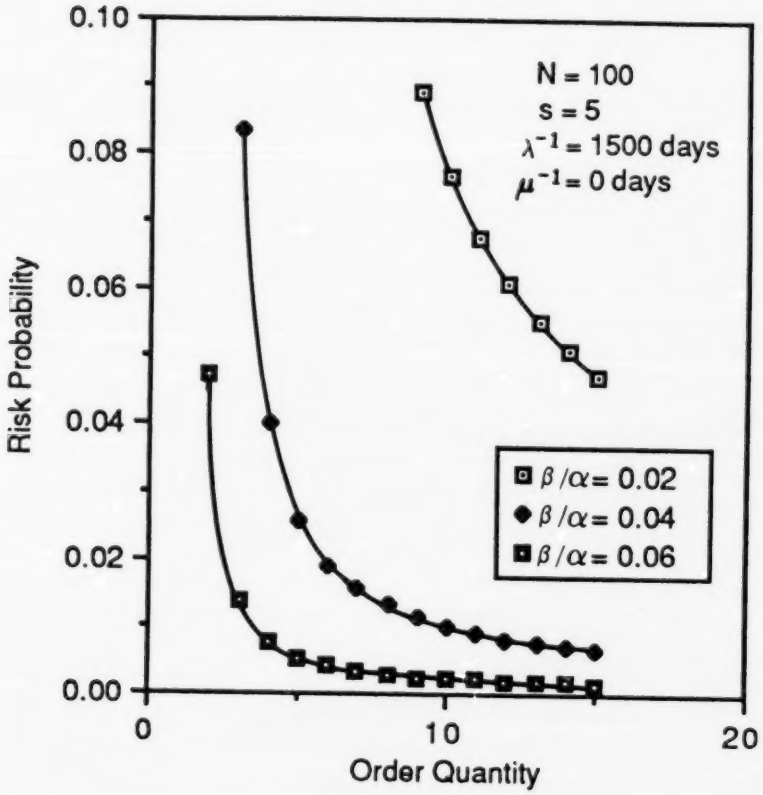

FIGURE 4

RISK PROBABILITY AS A FUNCTION OF ORDER QUANTITY
(indicating the effect of the ratio β/α)

CAZ DISK
SIM-FIG-4

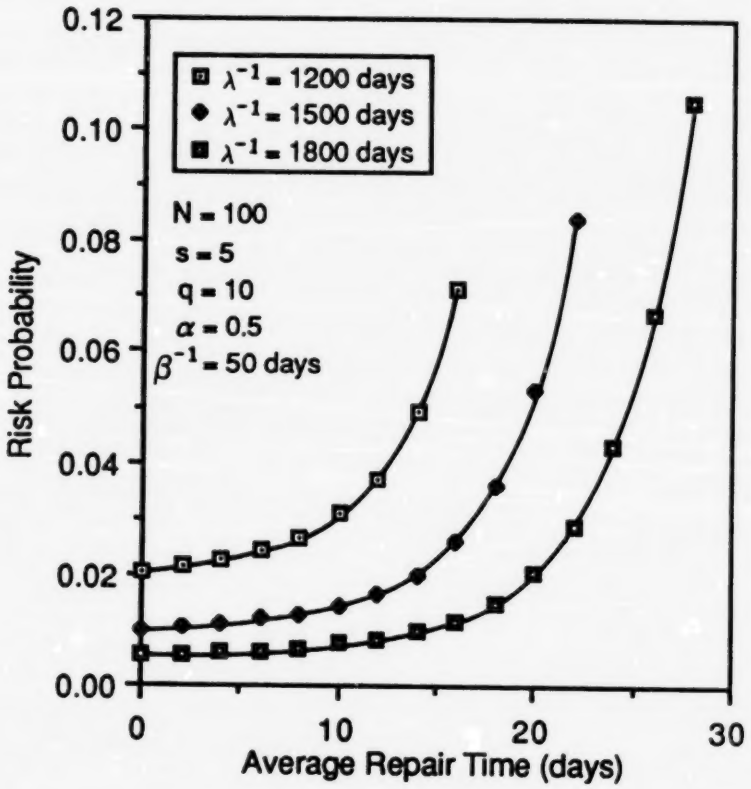

FIGURE 5

RISK PROBABILITY AS A FUNCTION OF AVERAGE REPAIR TIME

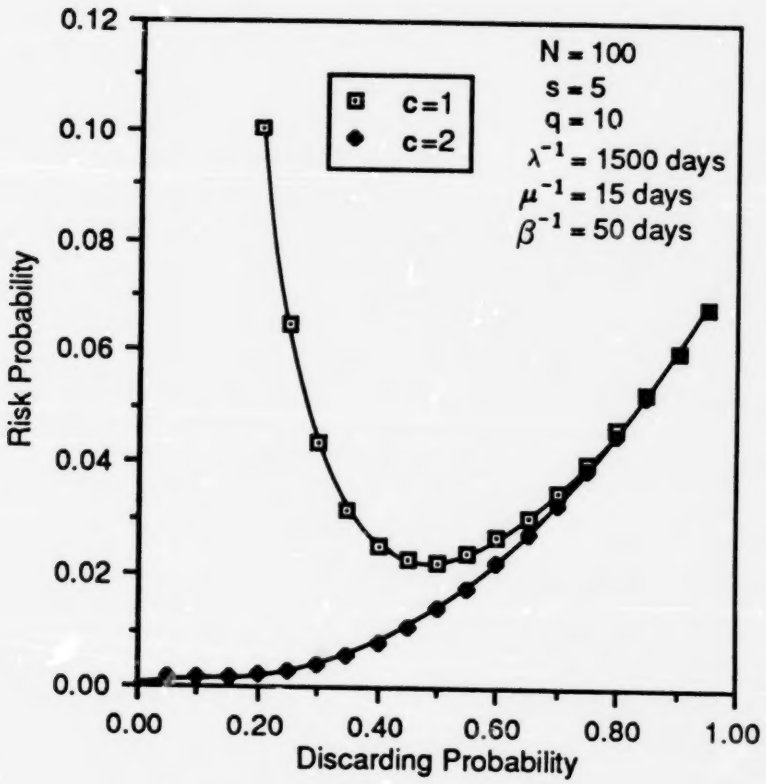

FIGURE 6

RISK PROBABILITY AS A FUNCTION OF DISCARDING PROBABILITY

(c = total number of repair teams)

CAZ DISK
SIM-FIG-6

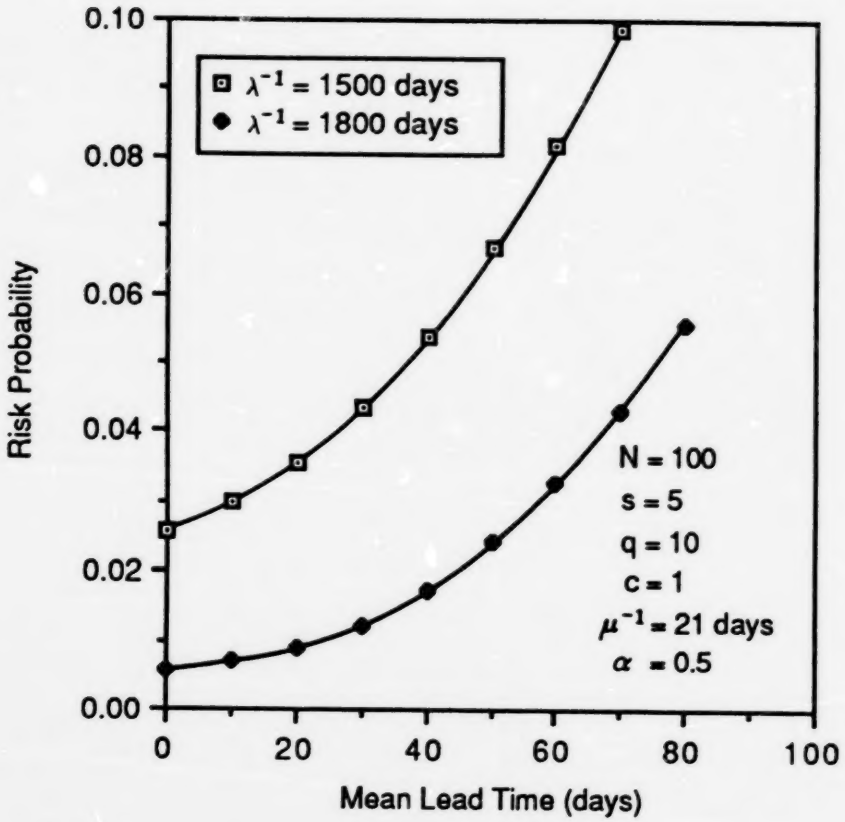

FIGURE 7

RISK PROBABILITY AS A FUNCTION OF MEAN LEAD TIME

CAZ DISK
SIM-FIG-7

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